Congruent Triangles

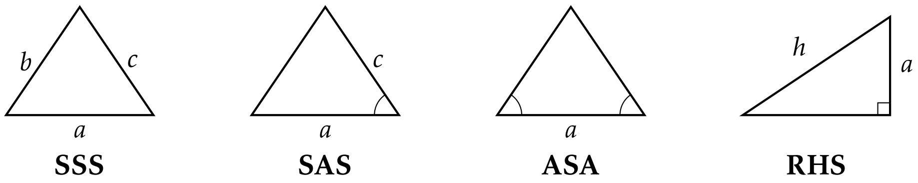
Things you should already know

Fact (Congruence) — Two shapes are **congruent** if they are exactly the same shape and size. Congruent shapes can be reflections or rotations of each other.

Fact (Angle Sum) — Angles in a triangle sum to 180°.

Fact — Two triangles are **congruent** if any one of the following conditions holds:

- **SSS** — all three pairs of sides are equal
- **SAS** — two pairs of sides and the **included** angle are equal
- **ASA** (or AAS) — two pairs of angles and a corresponding side are equal
- **RHS** — both have a right angle, equal hypotenuses, and one other equal side



Example
Why is **SSA** (two sides and a non-included angle) **not** sufficient for congruence?

*SSA can produce two different triangles — this is the **ambiguous case** of the sine rule. Given sides a, b and angle A (opposite a), the side b can "swing" to meet side a in two different positions, giving two non-congruent triangles with the same SSA information.*

Example

Why is **AAA** not sufficient for congruence?

AAA guarantees the same ***shape*** *but not the same* ***size****. The triangles could be enlargements of each other. AAA gives* ***similarity****, not congruence.*

Example

State the congruence condition for each pair:

(a) $\triangle ABC$ with $AB = 5$, $BC = 7$, $AC = 8$ and $\triangle DEF$ with $DE = 5$, $EF = 7$, $DF = 8$.

(b) $\triangle PQR$ with $PQ = 6$, $\angle Q = 90°$, $QR = 8$ and $\triangle XYZ$ with $XY = 6$, $\angle Y = 90°$, $YZ = 8$.

(c) $\triangle ABC$ with $\angle A = 40°$, $AB = 9$, $\angle B = 70°$ and $\triangle DEF$ with $\angle D = 40°$, $DE = 9$, $\angle E = 70°$.

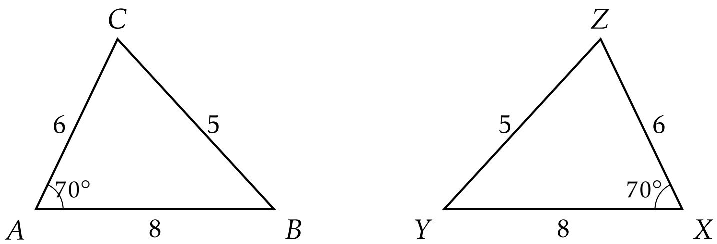
(a) SSS (all three sides match: $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$)

(b) SAS (two sides and the included right angle match)

(c) ASA (two angles and the included side match)

Example

Write corresponding vertices in the correct order.



$\triangle ABC \cong \triangle XYZ$ (SAS)
Corresponding vertices: $A \leftrightarrow X$ (both have the 70° angle), $B \leftrightarrow Y$, $C \leftrightarrow Z$.
We write $\triangle ABC \cong \triangle XYZ$ with vertices ***in matching order****.*

Similar Triangles

Example

A triangle has angles 40°, 60°, 80° and sides 5 cm, 6 cm, 7 cm. Another triangle has angles 40°, 60°, 80° and sides 10 cm, 12 cm, 14 cm. Are they congruent? What is special about them?

Not congruent (different sizes), but ***similar*** *— same shape, different size. All corresponding sides are in the ratio* 1 : 2 *(scale factor 2).*

Fact — Two triangles are **similar** if any one of the following conditions holds:

- **AAA** — two (and hence all three) pairs of angles are equal
- All three pairs of sides are **in the same ratio**
- Two pairs of sides are in the same ratio and the included angles are equal

If two triangles are similar, then all corresponding sides are in the same ratio (the **scale factor**).

Finding Unknown Lengths

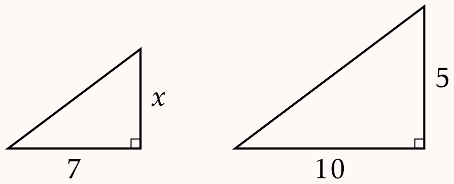
There are two methods:

Method 1 — Between triangles: compare corresponding sides across the two triangles.

Method 2 — Within triangles: compare sides within the same triangle.

Example

The triangles below are similar. Find x.



***Method 1** (between triangles):* $\frac{x}{5} = \frac{7}{10}$, *so* $x = 3.5$

***Method 2** (within triangles):* $\frac{x}{7} = \frac{5}{10}$, *so* $x = 3.5$

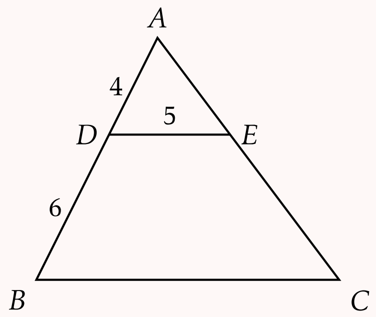
Spotting Similar Triangles

Fact — Look for similar triangles when you see:

- Parallel lines cutting through two sides of a triangle (by alternate/corresponding angles)
- Triangles sharing an angle, with other angles equal
- "Butterfly" configurations (two triangles sharing a vertex)

Example

In the diagram, DE is parallel to BC. $AD = 4$ cm, $DB = 6$ cm, $DE = 5$ cm. Find BC.



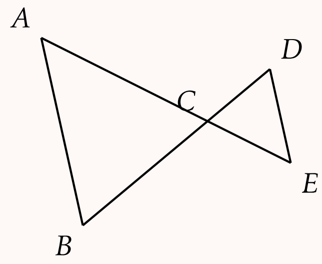
$\triangle ADE$ is similar to $\triangle ABC$ (AA: $\angle A$ is shared; $\angle ADE = \angle ABC$ by corresponding angles since $DE \parallel BC$).

Scale factor: $\frac{AB}{AD} = \frac{4+6}{4} = \frac{10}{4} = 2.5$

$BC = DE \times 2.5 = 5 \times 2.5 = 12.5$ *cm*

Example

In the diagram, BCD and ACE are straight lines. $AC = 12$, $BC = 8$, $CD = 6$, $CE = 4$. Show that $\triangle ABC$ is similar to $\triangle DEC$.



$\angle ACB = \angle DCE$ (vertically opposite angles).
Check ratios of sides adjacent to these angles:

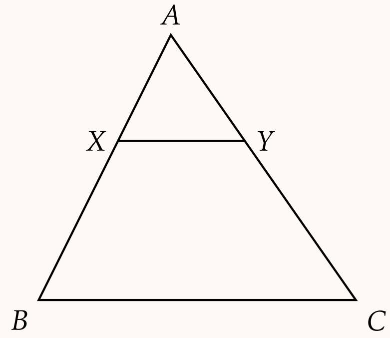
$$\frac{AC}{DC} = \frac{12}{6} = 2, \qquad \frac{BC}{EC} = \frac{8}{4} = 2$$

Two pairs of sides in the same ratio with equal included angles $\implies$ similar triangles.
$\triangle ABC \sim \triangle DEC$ (with scale factor 2).

Ratios in Similar Figures

Example

In the diagram, $XY \parallel BC$. $AX : XB = 2 : 3$. If $BC = 15$ cm, find XY.



$AX : XB = 2 : 3$, *so* $AX : AB = 2 : 5$.
$\triangle AXY \sim \triangle ABC$ *with scale factor* $\dfrac{AX}{AB} = \dfrac{2}{5}$.
$XY = \dfrac{2}{5} \times 15 = 6$ *cm.*